Higher-Order Linear-Time Unconditionally Stable Alternating Direction Implicit Methods for Nonlinear Convection-Diffusion Partial Differential Equation Systems

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We introduce a class of alternating direction implicit (ADI) methods, based on approximate factorizations of backward differentiation formulas (BDFs) of order $p \geq 2$, for the numerical solution of two-dimensional, time-dependent, nonlinear, convection-diffusion partial differential equation (PDE) systems in Cartesian domains. The proposed algorithms, which do not require the solution of nonlinear systems, additionally produce solutions of spectral accuracy in space through the use of Chebyshev approximations. In particular, these methods give rise to minimal artificial dispersion and diffusion and they therefore enable use of relatively coarse discretizations to meet a prescribed error tolerance for a given problem. A variety of numerical results presented in this text demonstrate high-order accuracy and, for the particular cases of $p = 2, 3$, unconditional stability. [DOI: 10.1115/1.4026868]

1 Introduction

We introduce a class of ADI methods, based on approximate factorizations of BDFs of order $p \geq 2$, for the numerical solution of two-dimensional time-dependent nonlinear convection-diffusion PDE systems in Cartesian domains. Similar to regular implicit time-marching methods, the algorithms proposed in this paper relax or altogether eliminate the Courant–Friedrichs–Lewy (CFL) stability constraints. Unlike previous implicit methods, however, the new approaches achieve unconditional stability without incurring the significant costs inherent in the nonlinear solvers associated with the nonlinear convective terms. Additionally, they produce solutions with high-order accuracy in space and time. Thus, these methods, which do not require the addition of numerical dissipation, give rise to reduced artificial dispersion and diffusion and, therefore, they enable reductions on the sizes of the discretizations required to meet a prescribed error tolerance for a given problem. To our knowledge, these are the first spatially high-order algorithms in the literature for which unconditional stability has been verified (if not rigorously proved) that exhibit, at the same time, high-order accuracy ($p = 2, 3$) in time without recourse to iterative solutions of nonlinear equations. (The well-known reference [1] presents an ADI algorithm of second-order accuracy in time and space which, relying on the use of numerical dissipation, enjoys unconditional stability for two-dimensional problems; the more recent contribution [2], in turn, achieves the same temporal accuracy but at the expense of the iterative solution of the nonlinear factored equations.) Algorithms of even higher temporal accuracy ($p \geq 4$) with modest CFL constraints are also presented in this text, which could be of significant interest in certain contexts. All of these approaches are developed in conjunction with both finite-difference and spectral spatial discretizations; in all cases, the appropriate orders of temporal accuracy are verified and unconditional stability ($p \leq 3$) is demonstrated.

(The use of fine spatial resolutions, which are often required to adequately represent complex domains with fine geometric features, boundary layers, turbulent solutions, etc., impose stringent numerical stability conditions for explicit time-marching methods; this effect is most pronounced for problems that include spatial diffusion. Implicit time-marching methods which, like the ones presented in this paper, can relax or altogether eliminate such numerical stability constraints, often do so at the expense of high computing costs. Indeed, a typical implicit step requires the inversion of a large generally nonlinear system of equations which, in multiple dimensions, can be very costly in terms of computation and memory requirements. In contrast, the alternating direction implicit methods we use enjoy the enhanced stability inherent in regular implicit methods but they do so at reduced computing costs. Methods that can ensure high-order accuracy, both in time and in space, on the other hand, give rise to reduced artificial dispersion and diffusion and they therefore enable reductions on the sizes of the discretizations required to meet a prescribed error tolerance.)

In contrast to other implicit methods, which must solve a multidimensional system of equations at every time step, an ADI algorithm evolves the solution of a multidimensional PDE one dimension at a time through the solution of a series of one-dimensional boundary-value problems. The present algorithms achieve high orders of temporal accuracy by means of novel factorizations of expressions resulting from the BDF time discretizations. The ordinary differential equation (ODE) system solver framework we introduce for the one-dimensional ADI problems can be used in conjunction with various spatial approximations, including Chebyshev polynomials, Fourier continuation (FC) [3–5], and finite differences. The implementation of the proposed ADI schemes, furthermore, is completely straightforward. For the sake of brevity, most of our examples concern the well-established Chebyshev spectral discretizations. Similar stability

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properties were observed in preliminary tests for spatial discretizations based on finite differences and the Fourier-continuation method [3,4]. (Preliminary tests indicate that, while the FC-based solver requires somewhat finer discretizations for a given accuracy, it also gives rise to smaller numbers of GMRES iterations than the Chebyshev-based method in connection with certain necessary variable-coefficient ODE solvers described in Sec. 4. The low-order finite-difference methods are, of course, much less efficient than either the Cheybshev- or FC-based algorithms.)

The derivation of our ADI schemes for a nonlinear PDE system relies on a few key observations. Most importantly, using the solution at time levels previous to $t = t^{n+1}$, the algorithm converts the nonlinear spatial operator into an implicit but linear operator with variable coefficients. The resulting approximately-factored equation is solved in "sweeps" along each of the Cartesian directions, including, as is common in ADI approaches, an intermediate "$t^{n+1/2}$" step. All of the proposed algorithms are embodied in a single formula that includes BDF-based ADI methods of temporal orders $p = 1, \ldots, 5$. While this paper has focused on the Burgers system, we note that, since the stability regions of all BDF methods contain the entire negative real axis, the methods should, more generally, be well-suited to a variety of problems with strong diffusive terms. However, the stability regions of BDF methods of order three and above omit portions of the imaginary axis and, thus, one might expect a hyperboliclike CFL condition for advection-dominated equations.

2 ADI Methods Based on Backward Differentiation Formulas

We derive a family of ADI methods of temporal orders as high as five for the numerical solution of time-dependent two-dimensional nonlinear convection-diffusion systems of PDEs in Cartesian domains. Although our ADI methods are based on BDFs, which are implicit methods for the numerical integration of ordinary differential equations, a similar strategy can, in principle, be used to derive ADI methods starting from other numerical ODE integration schemes. Our presentation begins with a brief review of BDF time-stepping methods for the numerical solution of systems of ODEs.

2.1 BDF Methods. BDFs are implicit multistep methods (see Ref. [6], p.492) for the numerical solution of the initial-value problems of the form

$$\begin{cases} y'(t) = f(t, y(t)), & t \in (0, T] \\ y(0) = y_0 \end{cases} \tag{1}$$

where $f(t, y)$ is a given real-valued (scalar or vector) function in $C((0, T] \times \mathbb{R}^d)$, $T > 0$ and where d is a positive integer. Letting $\Delta t > 0$ and partitioning the integration interval $I := (0, T]$ into subintervals $I_n := (t^n, t^{n+1}]$ for $n = 0, \ldots, N_{\Delta t}$, where $t^n = t^0 + n\Delta t$, a BDF method of order p approximates the value of $y'(^{n+1})$, using the first derivative of the interpolating polynomial passing through $p+1$ solution data-points at times t^{n+1}, $t^n, \ldots, t^{n-p+1}$. The corresponding time-stepping algorithm takes the form

$$y^{n+1} = \sum_{m=0}^{p-1} a_m y^{n-m} + b\Delta t f^{n+1} + O(\Delta t^{p+1}) \tag{2}$$

Table 1 displays the BDF coefficients for $p = 1, \ldots, 6$.

Table 1 Coefficients of BDF methods of order p for $p = 1, \ldots, 6$

p	a_0	a_1	a_2	a_3	a_4	a_5	b
1	1	0	0	0	0	0	1
2	$\frac{4}{3}$	$-\frac{1}{3}$	0	0	0	0	$\frac{2}{3}$
3	$\frac{18}{11}$	$-\frac{9}{11}$	$\frac{2}{11}$	0	0	0	$\frac{6}{11}$
4	$\frac{48}{25}$	$-\frac{36}{25}$	$\frac{16}{25}$	$-\frac{3}{25}$	0	0	$\frac{12}{25}$
5	$\frac{300}{137}$	$-\frac{300}{137}$	$\frac{200}{137}$	$-\frac{75}{137}$	$\frac{12}{137}$	0	$\frac{60}{137}$
6	$\frac{360}{147}$	$-\frac{450}{147}$	$\frac{400}{147}$	$-\frac{225}{147}$	$\frac{72}{147}$	$-\frac{10}{147}$	$\frac{60}{147}$

2.2 Two-Dimensional Burgers System. This section introduces a class of BDF-based ADI methods for the nonlinear Burgers system

$$\begin{cases} u_t + 2uu_x + vu_y + uv_y = \nu\Delta u + f_u(x,y,t), & (x,y,t) \in U \times (0,T] \\ v_t + vu_x + uv_x + 2vv_y = \nu\Delta v + f_v(x,y,t), & (x,y,t) \in U \times (0,T] \\ u(x,y,t) = g_u(x,y,t), v(x,y,t) = g_v(x,y,t), & (x,y,t) \in \partial U \times (0,T] \\ u(x,y,0) = u_0(x,y), v(x,y,0) = v_0(x,y), & (x,y) \in U \end{cases} \tag{3}$$

where $\nu > 0$ and $U \subset \mathbb{R}^2$ is a Cartesian domain. The final time $T > 0$, initial functions $\mathbf{u}_0(x,y) := (u_0(x,y), v_0(x,y))^T$, source functions $\mathbf{f}(x,y,t) := (f_u(x,y,t), f_v(x,y,t))^T$, and boundary-value functions $\mathbf{g}(x,y,t) := (g_u(x,y,t), g_v(x,y,t))^T$ are assumed to be given. For clarity, we re-express the system (3) in the vector form

$$\mathbf{u}_t + \mathcal{A}_1(\mathbf{u})\mathbf{u} + \mathcal{B}_1(\mathbf{u})\mathbf{u} = \nu\mathcal{A}_2\mathbf{u} + \nu\mathcal{B}_2\mathbf{u} + \mathbf{f} \tag{4}$$

where $\mathbf{u} = (u, v)^T$ and where

$$\mathcal{A}_1(\mathbf{u}) = \begin{pmatrix} 2u\partial_x & 0 \\ v\partial_x & u\partial_x \end{pmatrix}, \quad \mathcal{B}_1(\mathbf{u}) = \begin{pmatrix} v\partial_y & u\partial_y \\ 0 & 2v\partial_y \end{pmatrix} \tag{5a}$$

$$\mathcal{A}_2 = \begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx} \end{pmatrix}, \quad \mathcal{B}_2 = \begin{pmatrix} \partial_{yy} & 0 \\ 0 & \partial_{yy} \end{pmatrix} \tag{5b}$$

We will sometimes simply write $\mathcal{A}_1$ and $\mathcal{B}_1$ instead of $\mathcal{A}_1(\mathbf{u})$ and $\mathcal{B}_1(\mathbf{u})$, respectively. Applying a BDF formula of order p to Eq. (4) we obtain the semidiscrete system

$$\mathbf{u}^{n+1} = \sum_{m=0}^{p-1} a_m \mathbf{u}^{n-m} + b\Delta t[-\mathcal{A}_1(\mathbf{u}^{n+1}) - \mathcal{B}_1(\mathbf{u}^{n+1}) + \nu\mathcal{A}_2 + \nu\mathcal{B}_2]\mathbf{u}^{n+1} + b\Delta t\mathbf{f}^{n+1} + O(\Delta t^{p+1})$$

i.e.

$$[I + b\Delta t(\mathcal{A}_1(\mathbf{u}^{n+1}) - \nu\mathcal{A}_2) + b\Delta t(\mathcal{B}_1(\mathbf{u}^{n+1}) - \nu\mathcal{B}_2)]\mathbf{u}^{n+1} = \sum_{m=0}^{p-1} a_m \mathbf{u}^{n-m} + b\Delta t\mathbf{f}^{n+1} + O(\Delta t^{p+1}) \tag{6}$$

where I denotes the identity operator.

To avoid the requirement of a nonlinear solver in our algorithms we use approximations, of certain orders of accuracy q, of the solution at time t^{n+1} that result from the extrapolation of *known* solution values at previous time levels. The extrapolatory approximations we utilize for $\mathbf{u}^{n+1}$ are given by equations of the form

$$\tilde{\mathbf{u}}_q^{n+1} := \sum_{l=0}^{q-1} c_\ell \mathbf{u}^{n-l} \tag{7}$$

Table 2 Extrapolation coefficients c_ℓ for $\mathbf{u}^{n+1} = \sum_{\ell=0}^{q-1} c_\ell \mathbf{u}^{n-\ell} + O(\Delta t^q)$

q	c_0	c_1	c_2	c_3	c_4
2	2	−1	0	0	0
3	3	−3	1	0	0
4	4	−6	4	−1	0
5	5	−10	10	−5	1

where c_ℓ are coefficients that arise from the consideration of a polynomial interpolating $\{\mathbf{u}^n, \ldots, \mathbf{u}^{n-q+1}\}$ —so that, in particular, $\tilde{\mathbf{u}}_q^{n+1} = \mathbf{u}^{n+1} + O(\Delta t^q)$ as $\Delta t \to 0$. Table 2 contains the coefficients for an order q approximation of $\mathbf{u}^{n+1}$, for $q = 2, 3, 4, 5$.

By substituting $\mathbf{u}^{n+1}$ with $\tilde{\mathbf{u}}_p^{n+1}$ in the differential operators $\mathcal{A}_1$ and $\mathcal{B}_1$ in Eqs. (5*a*) and (5*b*), we obtain the approximations

$$\tilde{\mathcal{A}}_1 := \mathcal{A}_1(\mathbf{u}_p^{n+1}) \approx \mathcal{A}_1(\mathbf{u}^{n+1}) \quad \text{and} \quad \tilde{\mathcal{B}}_1 := \mathcal{B}_1(\tilde{\mathbf{u}}_p^{n+1}) \approx \mathcal{B}_1(\mathbf{u}^{n+1})$$

in terms of the *linear* differential operators $\tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{B}}_1$ with variable coefficients. Using these approximations in Eq. (6), we obtain a linear problem for $\mathbf{u}^{n+1}$. Clearly, this substitution preserves the formal order of temporal accuracy inherent in the previous temporal discretization (6).

Next, we use the identity

$$\begin{aligned}
&\left[I + b\Delta t(\tilde{\mathcal{A}}_1 - \nu\mathcal{A}_2)\right]\left[I + b\Delta t(\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2)\right]\mathbf{u}^{n+1} \\
&\quad = \left[I + b\Delta t(\tilde{\mathcal{A}}_1 - \nu\mathcal{A}_2) + b\Delta t(\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2)\right]\mathbf{u}^{n+1} \\
&\qquad + (b\Delta t)^2\left[\tilde{\mathcal{A}}_1 - \nu\mathcal{A}_2\right]\left[\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2\right]\mathbf{u}^{n+1}
\end{aligned} \tag{8}$$

to obtain an approximate factorization of Eq. (6); note that, because of the existing variable coefficients, the spatial operators in Eq. (8), in general, do not commute. Adding $(b\Delta t)^2[\tilde{\mathcal{A}}_1 - \nu\mathcal{A}_2][\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2]\mathbf{u}^{n+1}$ to both sides of Eqs. (6) and using Eq. (8), we find

$$\begin{aligned}
&\left[I + b\Delta t(\tilde{\mathcal{A}}_1 - \nu\mathcal{A}_2)\right]\left[I + b\Delta t(\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2)\right]\mathbf{u}^{n+1} \\
&\quad = \sum_{m=0}^{p-1} a_m \mathbf{u}^{n-m} + b\Delta t\mathbf{f}^{n+1} + O(\Delta t^{p+1}) \\
&\qquad + (b\Delta t)^2\left[\tilde{\mathcal{A}}_1 - \nu\mathcal{A}_2\right]\left[\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2\right]\mathbf{u}^{n+1}
\end{aligned} \tag{9}$$

We make an additional approximation of $\mathbf{u}^{n+1}$, this time of order $p-1$, which we denote by $\hat{\mathbf{u}}^{n+1}$ (not to be confused with $\tilde{\mathbf{u}}^{n+1}$), on the right-hand side of Eq. (9); we obtain

$$\begin{aligned}
&\left[I + b\Delta t(\tilde{\mathcal{A}}_1 - \nu\mathcal{A}_2)\right]\left[I + b\Delta t(\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2)\right]\mathbf{u}^{n+1} \\
&\quad = \sum_{m=0}^{p-1} a_m \mathbf{u}^{n-m} + b\Delta t\mathbf{f}^{n+1} + O(\Delta t^{p+1}) \\
&\qquad + (b\Delta t)^2\left[\tilde{\mathcal{A}}_1 - \nu\mathcal{A}_2\right]\left[\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2\right]\hat{\mathbf{u}}^{n+1}
\end{aligned} \tag{10}$$

note that the $(p-1)$-order approximation we use for $\mathbf{u}^{n+1}$ is sufficient in this context to maintain the extant order of accuracy.

Dropping terms of order Δt^{p+1} and higher we obtain the implicit (factored) time-marching scheme

$$\begin{aligned}
&\left[I + b\Delta t(\tilde{\mathcal{A}}_1 - \nu\mathcal{A}_2)\right]\left[I + b\Delta t(\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2)\right]\mathbf{v}^{n+1} \\
&\quad = \sum_{m=0}^{p-1} a_m \mathbf{v}^{n-m} + b\Delta t\mathbf{f}^{n+1} + (b\Delta t)^2\left[\tilde{\mathcal{A}}_1 - \nu\mathcal{A}_2\right]\left[\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2\right]\hat{\mathbf{v}}^{n+1}
\end{aligned}$$

which we express in the ADI form

$$\begin{cases}
\left[I + b\Delta t(\tilde{\mathcal{A}}_1 - \nu\mathcal{A}_2)\right]\mathbf{v}^{n+1/2} = \displaystyle\sum_{m=0}^{p-1} a_m \mathbf{v}^{n-m} + b\Delta t(-\tilde{\mathcal{B}}_1 + \nu\mathcal{B}_2)\hat{\mathbf{v}}^{n+1} + b\Delta t\mathbf{f}^{n+1} \\
\left[I + b\Delta t(\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2)\right]\mathbf{v}^{n+1} = \displaystyle\sum_{m=0}^{p-1} a_m \mathbf{v}^{n-m} + b\Delta t(-\tilde{\mathcal{A}}_1 + \nu\mathcal{A}_2)\mathbf{v}^{n+1/2} + b\Delta t\mathbf{f}^{n+1}
\end{cases} \tag{11}$$

This is our p-th order alternating-direction BDF algorithm, which we denote by BDF(p)-ADI. Note that each sweep in Eq. (11) requires the solution of a system of variable coefficient ODEs.

3 Boundary Conditions for ADI Schemes

The ADI schemes derived in Sec. 2 require the solution of a system of ODEs in each sweep, which must be supplemented with a proper set of boundary conditions to complete a properly posed boundary-value problem (BVP). The prescription of boundary conditions for $\mathbf{v}^{n+1}$ in the second sweep of Eq. (11) does not present a problem: we have $\mathbf{v} = \mathbf{g}$ on the boundary of U. Although the solution at the intermediate step is commonly labeled $\mathbf{v}^{n+1/2}$, on the contrary, this quantity does not approximate, in general, the solution at time $t^{n+1/2} = t^n + \Delta t/2$ with the appropriate order of accuracy: using this for the $\mathbf{v}^{n+1/2}$ boundary values given by $\mathbf{g}(t^{n+1/2})$ would, in general, degrade the order of accuracy of the overall solver.

To obtain consistent boundary conditions for $\mathbf{v}^{n+1/2}$ we use the ADI scheme itself. Starting from Eq. (11), we cross-add the left-hand side and right-hand side terms and simplify to obtain

$$\mathbf{v}^{n+1/2} = \mathbf{v}^{n+1} + b\Delta t\left[\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2\right]\left(\mathbf{v}^{n+1} - \hat{\mathbf{v}}^{n+1}\right) \tag{12}$$

Along the domain boundary Eq. (12) becomes

$$\mathbf{v}^{n+1/2} = \mathbf{g}^{n+1} + b\Delta t\left[\tilde{\mathcal{B}}_1 - \nu\mathcal{B}_2\right]\left(\mathbf{g}^{n+1} - \hat{\mathbf{g}}^{n+1}\right) \tag{13}$$

which simplifies further to $\mathbf{v}^{n+1/2} = \mathbf{g}$ if the boundary functions $\mathbf{g}$ are time-independent. Without proof we note that, even for time-dependent boundary values $\mathbf{g}$, these boundary conditions achieve the desired order of accuracy. This fact, which is clearly demonstrated in Fig. 2, can be established by considering the boundary-layer character of the error—which, for Δt small enough, gives rise to an additional exponentially small factor in the error arising from the boundary condition. Furthermore, convergence of order p for arbitrary values of Δt can be achieved by using approximations of order p instead of order $p-1$ in Eq. (10).

4 Numerical Solution of ODE Systems With Variable Coefficients

Each one of the two equations in Eq. (11) requires the solution of a second-order variable-coefficient system of the ODE of the form

$$\begin{cases} L\mathbf{u}(x) := P_2(x)\mathbf{u}''(x) + P_1(x)\mathbf{u}'(x) + P_0(x)\mathbf{u}(x) = \mathbf{r}(x), & -1 < x < 1, \\ \mathbf{u}(-1) = \mathbf{u}_l, \mathbf{u}(1) = \mathbf{u}_r \end{cases} \tag{14}$$

for $\mathbf{u} := (u_1(x), u_2(x))^T$ over the interval $[-1, 1]$, where $\mathbf{u}_l := (u_{1,l}, u_{2,l})^T$ and $\mathbf{u}_r := (u_{1,r}, u_{2,r})^T$ are constant vectors. It is easy to check that the system (14) possesses a unique solution. In our algorithm, the solution to Eq. (14) is approximated by means of a Chebyshev grid $\{x_j\}_{j=1}^N \subset [-1, 1]$, where $x_j = -\cos(\pi(j-1)/(N-1))$.

Let V_N be a collection of discrete functions defined at the grid points x_j for $j = 1, \ldots, N$. Letting $\mathbf{v}_N = (v_{1,N}, v_{2,N})^T \in V_N^2$, we will occasionally write $\mathbf{v}_j$ instead of $\mathbf{v}_N(x_j)$ for brevity. We note that the approximation of Eq. (14) can be succinctly restated as follows: find $\mathbf{v}_N \in V_N^2$, such that

$$\begin{cases} L_N \mathbf{v}_N(x_j) = \mathbf{r}(x_j), & j = 2, \ldots, N-1, \\ \mathbf{v}_1 = \mathbf{u}_l, \mathbf{v}_N = \mathbf{u}_r \end{cases} \tag{15}$$

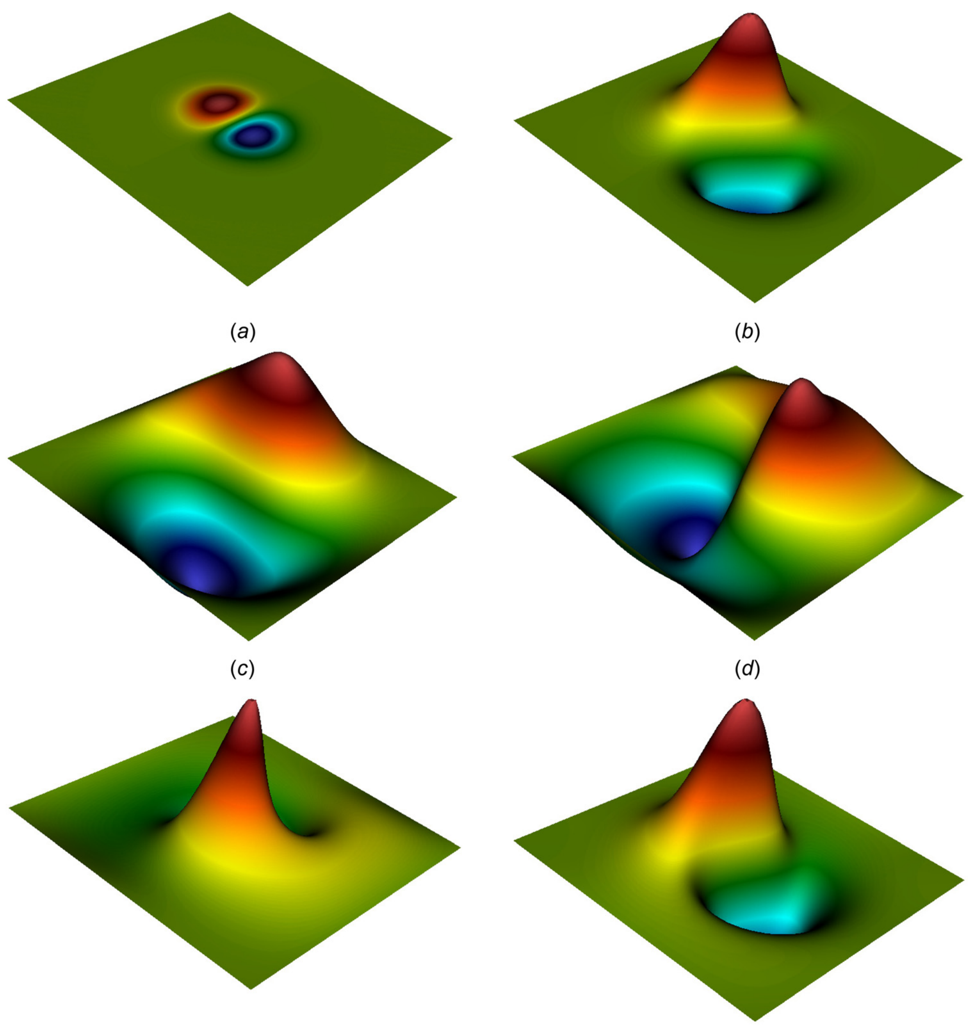
where L_N, which is a discrete version of the operator L defined in Eq. (14), is defined as

$$L_N \mathbf{v}(x_j) := P_2(x_j)\mathbf{v}^{(2)}(x_j) + P_1(x_j)\mathbf{v}^{(1)}(x_j) + P_0(x_j)\mathbf{v}(x_j), \quad j = 2, \ldots, N-1$$

The approximate first and second derivatives $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are evaluated as follows: letting $v_{1,N} = (v_{1,1}, \ldots, v_{1,N})^T$ and $v_{2,N} = (v_{2,1}, \ldots, v_{2,N})^T$ and letting $D = [d_{jk}], j, k = 1, \ldots, N$ denote the Chebyshev differentiation operator, then the mth derivative $v_{1,N}^{(m)}$ is computed as $v_{1,N}^{(m)} = D^m(v_{1,1}, \ldots, v_{1,N})^T$ and similarly for $v_{2,N}$. Finally, the linear system (15) is vectorized and solved using GMRES. To accelerate the convergence of the GMRES solver, we use standard second-order finite difference approximations to Eq. (14) (that also satisfy the boundary conditions) as preconditioners.

5 Numerical Results

In this section we present examples that illustrate the accuracy and stability properties of the BDF-ADI methods developed in this text. In these examples, spatial derivatives are evaluated by means of Chebyshev approximations, as described in Sec. 4, or, for comparison purposes and to demonstrate the generality of the methodology proposed in this paper, by means of finite differences of order two. Denoting by $I_N, \tilde{\mathcal{A}}_{1N}, \mathcal{A}_{2N}, \tilde{\mathcal{B}}_{1N}$, and $\mathcal{B}_{2N}$ the discrete approximations used for the spatial differential operators $I, \tilde{\mathcal{A}}_1, \mathcal{A}_2, \tilde{\mathcal{B}}_1$, and $\mathcal{B}_2$, respectively (resulting from, e.g., the Cheybshev differentiation, finite-differences, etc.), the boundary-value problems (11) take the fully discrete forms



(a) (b) (c) (d) (e) (f)

Fig. 1 Burgers system solution in a Cartesian domain using BDF(3)-ADI and Chebyshev approximations

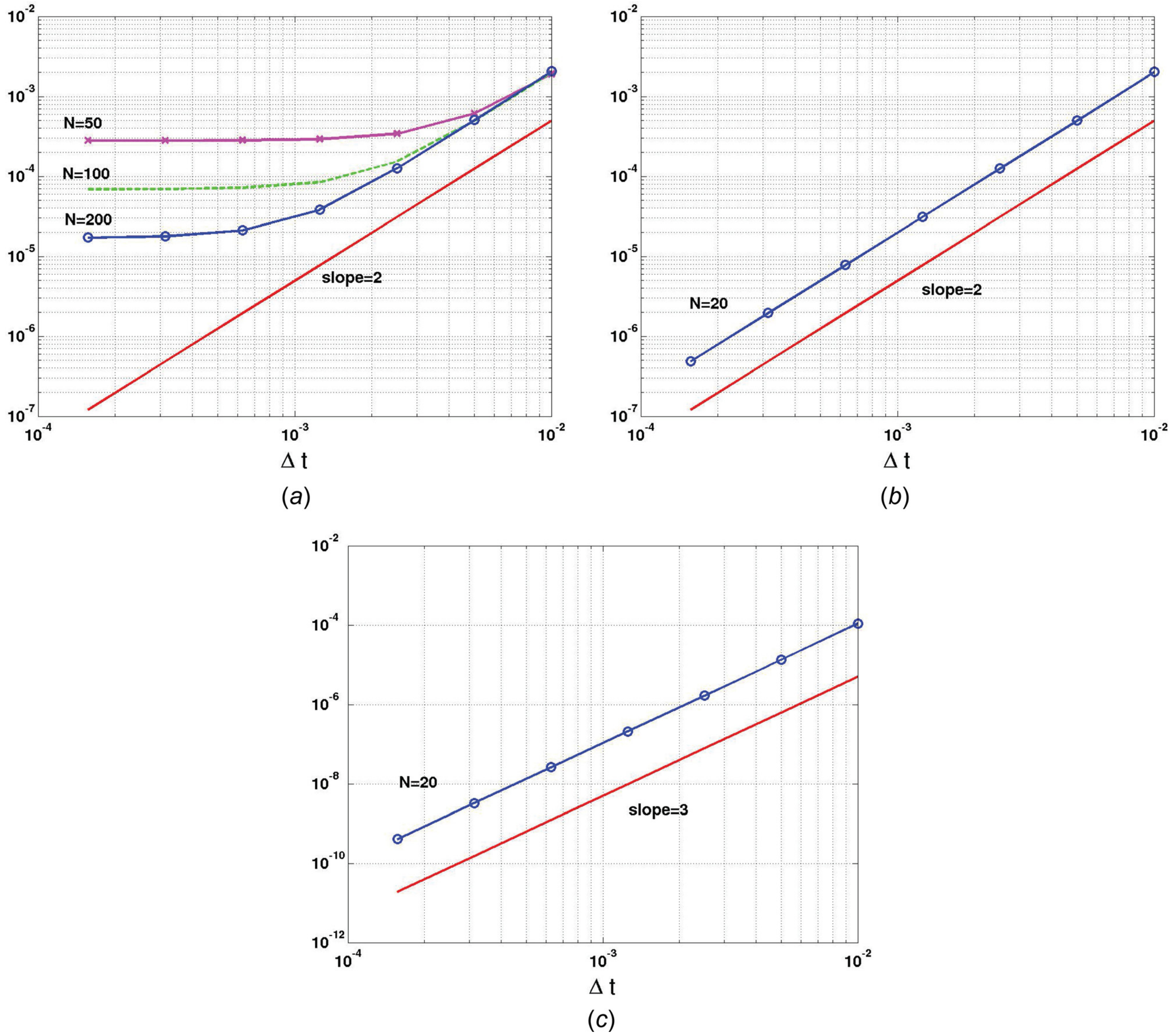


Fig. 2 Temporal convergence as $\Delta t \to 0$, using various spatial resolutions ($N_x = N_y = N$), of the approximate solution to the system (3) over $[0,1]^2$. Maximum errors versus the time step Δt are obtained by means of various methods. (*a*) second-order spatial finite differences with $N = 50, 100, 200$ and BDF(2)-ADI. (*b*) Chebyshev spatial approximation with $N = 20$ and BDF(2)-ADI. (*c*) Chebyshev spatial approximation with $N = 20$ and BDF(3)-ADI.

$$\left\{ \begin{aligned} &\left[I_N + b\Delta t(\tilde{\mathcal{A}}_{1N} - \nu\mathcal{A}_{2N})\right]\mathbf{v}_N^{n+1/2} = \sum_{m=0}^{p-1} a_m \mathbf{v}_N^{n-m} + b\Delta t(-\tilde{\mathcal{B}}_{1N} + \nu\mathcal{B}_{2N})\hat{\mathbf{v}}_N^{n+1} + b\Delta t\mathbf{f}^{n+1} \\ &\left[I_N + b\Delta t(\tilde{\mathcal{B}}_{1N} - \nu\mathcal{B}_{2N})\right]\mathbf{v}_N^{n+1} = \sum_{m=0}^{p-1} a_m \mathbf{v}_N^{n-m} + b\Delta t(-\tilde{\mathcal{A}}_{1N} + \nu\mathcal{A}_{2N})\mathbf{v}_N^{n+1/2} + b\Delta t\mathbf{f}^{n+1} \end{aligned} \right. \tag{16}$$

As mentioned in Sec. 4, these two boundary value problems (which correspond to the two discrete half-steps (11)) are solved by means of the iterative linear algebra solver GMRES.

Throughout this section, we take the number of points along each dimension of a Cartesian domain $U := [x_L, x_R] \times [y_B, y_T] \subset \mathbb{R}^2$, such that $N_x = N_y =: N$. In the Chebyshev case, the approximate solution $\mathbf{v}_N(x_j, y_k) = \mathbf{v}_{jk} \approx \mathbf{u}_{jk}$ is computed over (appropriately scaled and translated versions of) the Chebyshev grids $x_j = -\cos(\pi(j-1)/(N-1))$ and $y_k = -\cos(\pi(k-1)/(N-1))$. For the finite-difference examples, in turn, we utilize the uniform mesh $\{(x_j, y_k)\}_{j=1,k=1}^{N_x,N_y} \subset U$, where $x_j = x_L + (j-1)\Delta x$, $y_k = y_B + (k-1)\Delta y$ and $\Delta x = (x_R - x_L)/(N_x - 1)$, $\Delta y = (y_T - y_B)/(N_y - 1)$.

In our first example, we consider the system (3) over a Cartesian domain $U := [-3, 3]^2$. For the initial condition and boundary functions we use $\mathbf{u}_0(x, y) = (u_0(x, y), v_0(x, y))^T \equiv 0$ and $\mathbf{g}(x, y, t) = (g_u(x, y, t), g_v(x, y, t))^T \equiv 0$; the source terms, in turn, are set to $\mathbf{f} = (f_u(x, y, t), f_v(x, y, t))^T$ with

$$f_u(x, y, t) = Ae^{-r/\sigma^2}(y\cos(\theta) + x\sin(\theta)) \quad \text{and}$$
$$f_v(x, y, t) = Ae^{-r/\sigma^2}(y\sin(\theta) - x\cos(\theta))$$

where $r=\sqrt{x^2+y^2}$ and $\theta=t\pi/2$ and where $\sigma^2=0.4$ and $A=100$. The solution of this problem is the rotating vortex depicted in Fig. 1. This solution was obtained using $N=100$ and $\Delta t=10^{-2}$ for various times t in the interval $t\in(0,5]$. The solution was computed by means of the BDF(3)-ADI method in conjunction with Chebyshev spatial differentiation operators. These images demonstrate the stability of the algorithm, despite the extremely small minimum Chebyshev mesh-size (which is on the order of 10^{-4}): for an explicit method the CFL constraint (time step on the order of the square of the minimum mesh-size) requires $\Delta t \lesssim 10^{-8}$. In the present method, however, we see that stability (with high-order accuracy) results from use of the time step $\Delta t=10^{-2}$—without recourse to the solution of challenging nonlinear systems, which are associated with classical implicit solvers.

In order to easily quantify errors, in our second example we consider a problem involving a manufactured solution $\mathbf{u}=(u,v)^T$ of 3 over $U=[0,1]^2$ given by

$$u(x,y,t)=\sin(2\pi k_x(x+t))\sin(2\pi k_y(y+t)) \tag{17a}$$

$$v(x,y,t)=\sin(2\pi k_x(x+t))+\sin(2\pi k_y(y+t)) \tag{17b}$$

where $k_x=k_y=1$ and the corresponding source terms $\mathbf{f}$ and Dirichlet boundary conditions. To verify the expected temporal order of accuracy in Fig. 2 we present the maximum error

$$\|\mathbf{u}-\mathbf{v}\|_{\max}=\max_{0\le i,j\le N}\{|u_{ij}-v_{ij}|\}$$

at the final time $T=0.01$ versus a range of time step sizes Δt and several values of N. The derivatives and ODE systems are approximated using centered second-order finite differences (FD-2) and Chebyshev approximations. The results indicate that BDF(2)-ADI and BDF(3)-ADI achieve second-order and third-order temporal accuracy, as expected.

To conclude this section we use the solution (17) over $[0,1]^2$ to demonstrate the unconditional stability of the proposed BDF-ADI schemes. We fix $\nu=1$, the final time $T=1000$, and the number of points $N=20$ and solve the system (16) for time step sizes $\Delta t=1, 10^{-1}$, and 10^{-2} with BDF-ADI(2) and BDF-ADI(3). Chebyshev approximations are used in all cases. Figure 3 shows that while the solution is, in some cases, inaccurate (as it should be, in view of the extremely coarse time-steps used), the maximum error remains bounded. Note that a typical explicit time-marching scheme coupled with a Chebyshev approximation

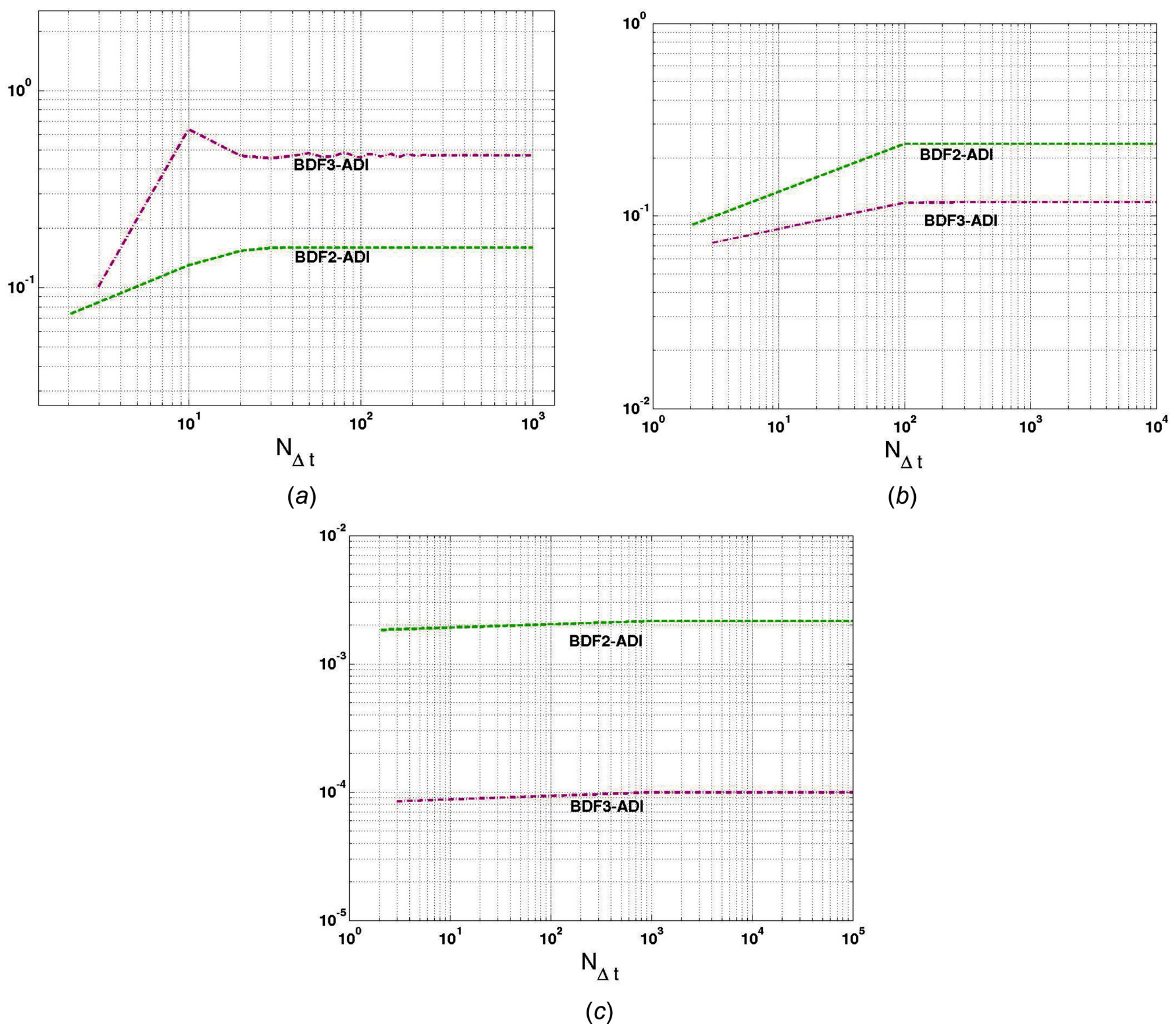


Fig. 3 Stability of the BDF(2)-ADI and BDF(3)-ADI temporal schemes when used in conjunction with the Chebyshev spatial approximations, as demonstrated by the display of the maximum error at a final time $T=1000$ with a fixed spatial resolution $N=20$ and various time-steps. (*a*) $\Delta t=1$. (*b*) $\Delta t=10^{-1}$. (*c*) $\Delta t=10^{-2}$.

would impose a stability constraint proportional to $1/N^4$ so that with $N = 20$, $\Delta t \approx 10^{-5}$ would be necessary for stability. The results presented in Fig. 3 demonstrate that the proposed algorithms remain stable for values of Δt that are orders of magnitude beyond the stability limit required by explicit methods.

6 Conclusions

We have presented a class of alternating direction implicit methods based on approximate factorizations of backward differentiation formulas of order p for the numerical solution of a two-dimensional nonlinear PDE system of Burgers equations in a Cartesian domain. Our ADI schemes can be coupled with various spatial approximations such as standard or compact finite differences, Fourier continuation, and Chebyshev approximations. Thus, by combining different BDF(p)-ADI schemes and spatial approximations, an overall algorithm can be devised that is high-order in both time and space or even spectral in space if Chebyshev approximations are used. Clearly, suitable modifications of the proposed approaches could be used to enable the solution of a variety of linear and nonlinear PDEs—including; for example, the Navier–Stokes equations.

As in the original linear Peaceman–Rachford method, our ADI schemes evolve the solution from one time-level to the next by means of the solution of a sequence of one-dimensional boundary-value problems. Unlike some recent ADI methods for linear problems [7], which require multiple fractional steps to achieve a high temporal order, our BDF(p)-ADI schemes utilize a single one-dimensional BVP solve per dimension and they do not require Richardson's extrapolation. The ODE system solver framework we presented, whose key ingredient is a preconditioned GMRES solver in the case of the dense matrices that result from Fourier continuation or Chebyshev approximations, or a banded linear system solver in the finite difference case, can be implemented in a straightforward manner.

Extensive numerical experiments indicate that the schemes BDF(2)-ADI and BDF(3)-ADI exhibit unconditional stability. On the contrary, while methods such as the BDF(4)-ADI and BDF(5)-ADI appear to be subject to a stability constraint similar to a CFL condition, they nevertheless remain stable with time step sizes that are orders of magnitude larger than the stability limit imposed by explicit time-marching schemes for equations with second-order derivatives. Of course, only a rigorous analysis of the underlying schemes can conclusively establish the true numerical stability limits of the methods presented; research in this regard is presently ongoing. While the ADI methods presented here were illustrated with initial-boundary value problems defined over Cartesian domains, the preliminary results suggest that these approaches can also be applied in complex curvilinear domains; such extensions, however, have been left for future work.

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